

SPANS AND SIMPLICIAL FAMILIES

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ABSTRACT. In [3] we have introduced the notion of *covering projection* on a general topos \mathcal{E} . These are locally constant objects with an additional property. We show there that the category $\mathcal{G}_{\mathcal{U}}$ of covering projections trivialized by a fix cover \mathcal{U} is an atomic topos with points. This determines a progroupoid of localic groupoids suitable indexed by a filtered poset of covers, which generalize the known results on the fundamental progroupoid of a locally connected topos to general topoi.

In this paper we consider simplicial families, that is, simplicial objects in \mathcal{E} indexed by a simplicial set. We show that covering projections can be defined as objects constructed from a descent datum of a simplicial set on a family of sets. The simplicial set is the index of a hypercover refinement of the cover. In particular, we show that any locally constant object in a locally connected topos is constructed by descent from a descent datum on a family of sets. We construct a groupoid $\mathbf{G}_{\mathcal{H}}$ such that the category $\mathcal{G}_{\mathcal{H}}$ of covering projections trivialized by a fix hypercover \mathcal{H} is its classifying topos. This determines a progroupoid of ordinary groupoids, this time suitable indexed by a filtered poset of hypercovers. Thus, by switching from covers to hypercovers we construct the fundamental progroupoid of a general topos as a progroupoid of ordinary groupoids. This construction is novel also in the case of locally connected topoi. The salient feature that distinguishes these topoi is that the progroupoid is *strict*, which is not the case in general.

CONTENTS

1. Spans	2
2. Simplicial sets	3
3. Simplicial families	5
4. Family hypercovers	8
5. Fundamental groupoid of a simplicial family	13
6. Descent	14
7. Covering projections	16
8. Fundamental progroupoid of a topos	18
References	19

PREFACE. Given a locally connected topos $\mathcal{E} \xrightarrow{\gamma} \mathbf{Set}$, the category $\mathcal{P}_{\mathcal{U}}$ of locally constant objects trivialized by a fix cover \mathcal{U} is the classifying topos of an ordinary groupoid $\mathbf{G}_{\mathcal{U}}$, $\mathcal{P}_{\mathcal{U}} = \beta \mathbf{G}_{\mathcal{U}}$. This determines a strict progroupoid $\pi_1(\mathcal{E})$ suitable indexed by a filtered poset of covers. This progroupoid is the fundamental progroupoid of \mathcal{E} in the sense that for any (discrete) group K it classifies K -torsors.

For non-locally connected topos the category $\mathcal{P}_{\mathcal{U}}$ is not any more the classifying topos of a groupoid. In [3] we have introduced the notion of *covering projection* (locally constant objects with an additional property) and show that the category $\mathcal{G}_{\mathcal{U}}$ of covering projections trivialized by a fix cover \mathcal{U} is an atomic topos with points. Then by the results of [6] it is the classifying topos of localic groupoid $\mathbf{G}_{\mathcal{U}}$, $\mathcal{G}_{\mathcal{U}} = \beta\mathbf{G}_{\mathcal{U}}$. This determines a progroupoid on localic groupoids $\pi_1(\mathcal{E})$ which generalize the result on the fundamental progroupoid of a locally connected topos to general topoi.

In this paper we consider simplicial families, that is, simplicial objects in \mathcal{E} indexed by a simplicial set, $H_{\bullet} \rightarrow \gamma^*(S_{\bullet})$. We show that covering projections can be defined as objects trivialized by a descent datum of a simplicial set S_{\bullet} on a family of sets $R \rightarrow S_0$. The simplicial set is the index of a suitable hypercover refinement $H_{\bullet} \rightarrow U_{\bullet}$ of the Čech simplicial family $U_{\bullet} \rightarrow \gamma^*(N_{\bullet})$ (where N_{\bullet} is the nerve of the cover). Since for a locally connected topos every locally constant object is a covering projection, it follow that in this case, using hypercovers instead of simple covers, *locally constant objects can be constructed by descent data on a family of sets*.

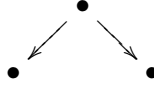
We construct an ordinary groupoid $\mathbf{G}_{\mathcal{H}}$ such that the category $\mathcal{G}_{\mathcal{H}}$ of covering projections trivialized by a fix hypercover \mathcal{H} is its classifying topos, $\mathcal{G}_{\mathcal{H}} = \beta\mathbf{G}_{\mathcal{H}}$. This determines a progroupoid of ordinary groupoids $\pi_1(\mathcal{E})$ suitable indexed this time by a filtered poset of hypercovers. Thus, by switching from covers to hypercovers we construct the fundamental progroupoid of a general topos as a progroupoid of ordinary groupoids. The salient feature that distinguishes the case of locally connected topos is that in that case the progroupoid is *strict*, that is, the transition morphisms are surjective on triangles (composably onto in the sense of [7]), and its classifying topos $\mathcal{G}(\mathcal{E}) = \beta\pi_1(\mathcal{E})$ is a Galois topos, features which do not hold in general.

I thank Matias de Hoyo for several fruitful conversations on the coskeleton construction.

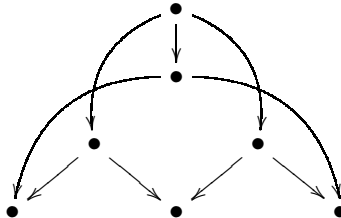
CONTEXT. Throughout this paper $\mathcal{S} = \mathbf{Sets}$ denotes the topos of sets. However, we argue in a way that should be valid if \mathcal{S} is an arbitrary Grothendieck topos, but let the interested reader to verify this.

1. SPANS

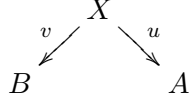
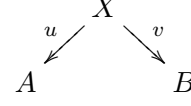
A *1-span* is a diagram of the form



A *2-span* is a commutative diagram of the form



The *dual* span is the span resulting from the symmetry respect the vertical axle. For example, the dual span of the span



In the same way we define the dual of a 2-span to be 2-span

resulting from the symmetry respect the vertical axle. We stress the fact that spans are *ordered from left to right* structures.

A *n-span* is the commutative diagram resulting from the following procedure: At the top vertex stands a generic n -simplex (see section 2), then draw an arrow to each of its $n + 1$ faces (considered ordered by the index). Then, repeat this procedure. At the bottom level stands the $n + 1$ vertices of the generic n -simplex.

2. SIMPLICIAL SETS

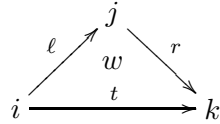
We fix some notation about simplicial sets, denoted by S_\bullet . A simplicial set has faces ∂_i and degeneracies σ_i as follows:

$$S_n \xrightleftharpoons[\sigma_i]{\partial_i} S_{n-1}, \quad n = 0, 1, \dots, \infty, \quad \{\partial_i\}_{0 \leq i \leq n}, \quad \{\sigma_i\}_{0 \leq i \leq n-1}$$

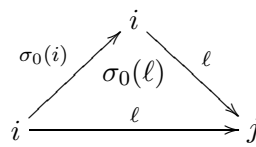
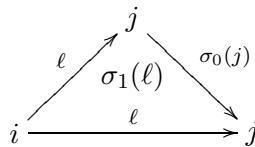
subject to the usual equations.

We write $I = S_0$, the set of vertices or 0-simplexes. Given a 1-simplex $\ell \in S_1$ and $i, j \in I$, we write $i \xrightarrow{\ell} j$ to mean $i = \partial_1(\ell)$, $j = \partial_0(\ell)$.

Given a 2-simplex $w \in S_2$, we write



to mean that $\partial_2(w) = \ell$, $\partial_1(w) = t$, $\partial_0(w) = r$. We define $i = \varrho_2(w)$, $j = \varrho_1(w)$, $k = \varrho_0(w)$ to be the three pairs of equal composites of faces. The simplicial equations show that this fits correctly. We say that the pair $i \xrightarrow{\ell} j \xrightarrow{r} k$ *compose*. Notice that we have:



Recall that a category can be seen as a simplicial set such that given any pair $i \xrightarrow{\ell} j \xrightarrow{r} k$ there is a unique $w \in S_2$ such that $\partial_2(w) = \ell$, $\partial_0(w) = r$.

We recall now the construction of a category and a groupoid associated to a simplicial set, which involve only the first three terms.

$$\begin{array}{ccccc}
 & \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & \\
 & \xleftarrow{\sigma_0} & & \xleftarrow{\sigma_0} & \\
 S_2 & \xrightarrow{\partial_1} & S_1 & \xleftarrow{\partial_1} & S_0 \\
 & \xleftarrow{\sigma_1} & & \xleftarrow{\partial_1} & \\
 & \xrightarrow{\partial_2} & & &
 \end{array}$$

2.1. Proposition (Fundamental category and groupoid of a simplicial set, [4]).

Objects: The set of objects is the set $I = S_0$.

Premorphisms: Basic premorphisms $i \xrightarrow{\ell} j$, $i, j \in S_0$, are 1-simplexes $\ell \in S_1$, $\partial_0(\ell) = j$, $\partial_1(\ell) = i$. A general premorphism $i \xrightarrow{\phi} j$ is a sequence $\phi = (\ell_n \dots \ell_2 \ell_1)$, $\ell_k \in S_1$, $\partial_1(\ell_1) = i$, $\partial_1(\ell_{k+1}) = \partial_0(\ell_k)$, $n \geq 1$, $1 \leq k \leq n-1$, $\partial_0(\ell_n) = j$. When $n = 1$ we write $(\ell_1) = \ell$. Premorphisms compose by concatenation.

Morphisms: The set of morphisms is the quotient of the set of premorphisms by the equivalent relation generated by the following basic pairs:

(1a) Given $i \xrightarrow{\ell} j \xrightarrow{r} k$, then $i \xrightarrow{(r\ell)} k \sim i \xrightarrow{t} k$ if there is $w \in S_2$ such that $\partial_2(w) = \ell$, $\partial_1(w) = t$, $\partial_0(w) = r$. That is, for each $w \in S_2$ we establish $\partial_1(w) \sim (\partial_0(w) \partial_2(w))$.

The arrow $i \xrightarrow{\sigma_0(i)} i$ becomes the identity morphism of i , $id_i = \sigma_0(i)$.

(1b) The groupoid is obtained by formally inverting all the arrows of the category. \square

2.2. Definition. A contravariant simplicial morphism $S_\bullet \xrightarrow{h_\bullet} T_\bullet$ between two simplicial sets is a family of maps $S_n \xrightarrow{h_n} T_n$ such that

$$\partial_i(h_n(w)) = h_{n-1}(\partial_{n-i}(w)), \quad \sigma_i(h_{n-1}(w)) = h_n(\sigma_{n-1-i}(w)).$$

2.3. Definition. A strict duality in a simplicial set S_\bullet is a contravariant simplicial isomorphism $S_\bullet \xrightarrow{\tau_\bullet} S_\bullet$ with $\tau_0 = id$. We denote τ in both directions $\tau \circ \tau^{-1} = id$. A simplicial set with a strict duality is said to be self-dual.

For any vertex i , $\tau_0(i) = i$. For any n -simplex w , $n > 0$, we will denote $\tau_n(w) = w^{op}$, omitting the n .

$$w \in S_n : \partial_i(w^{op}) = \partial_{n-i}(w)^{op}, \quad \sigma_i(w^{op}) = (\sigma_{n-1-i}(w))^{op}.$$

2.4. Remark. Clearly, the notion of strict duality applies to a simplicial object in any category.

The following is clear:

2.5. Proposition. Let S_\bullet be a self-dual simplicial set such that:

$$\forall i \xrightarrow{\ell} j \in S_1 \exists w \in S_2, \quad \begin{array}{ccc} & j & \\ \ell \nearrow & & \searrow \ell^{op} \\ i & \xrightarrow{\partial_1(w)} & i \end{array}, \quad \partial_1(w) = \sigma_0(i).$$

Then the fundamental category is already groupoid. \square

2.6. Example (Cech nerve). Given a family $\mathcal{U} = (U, I, \zeta)$, $\{U_i\}_{i \in I}$, $U \xrightarrow{\zeta} \gamma^* I$, in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, the *Cech simplicial set*¹ is the simplicial set N_\bullet whose n -simplexes are given by $N_n = \{(i_0, i_1, \dots, i_n) \mid U_{i_0} \times U_{i_1} \times \dots \times U_{i_n} \neq \emptyset\} \subset I^{n+1}$, in particular $N_0 = I$, $N_1 = \{(i, j) \mid U_i \times U_j \neq \emptyset\}$, $N_2 = \{(i, j, k) \mid U_i \times U_j \times U_k \neq \emptyset\}$.

The reader can check that it is a self-dual simplicial set. For $i \in N_0$, $w = (i, j, k) \in N_2$, $\sigma_0(i) = (i, i)$, and $\partial_2(w) = (i, j)$, $\partial_0(w) = (j, k)$, $\partial_1(w) = (i, k)$. Given $\ell = (i, j) \in N_1$, $\ell^{op} = (j, i)$. Then $w = (i, j, i)$ establish the condition in proposition 2.5. Thus the fundamental category is a groupoid. \square

3. SIMPLICIAL FAMILIES

Recall that a family in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ is an arrow $\zeta: H \rightarrow \gamma^* S$. In alternative notation we write $\mathcal{H} = \{H_i\}_{i \in S}$. We say that the objects H_i are the *components* of H . Families are 3-tuples $\mathcal{H} = (H, S, \zeta)$, and H is the coproduct $H = \sum_{i \in S} H_i$ in \mathcal{E} .

Remark that the same object H can be indexed by a different set, having then a different set of components.

A *morphism of families* $(Y, J, \xi) \xrightarrow{(h, \alpha)} (H, S, \zeta)$, is a pair $Y \xrightarrow{h} H$, $J \xrightarrow{\alpha} S$, making the following square commutative:

$$\begin{array}{ccc} Y & \xrightarrow{h} & H \\ \downarrow \xi & & \downarrow \zeta \\ \gamma^* J & \xrightarrow{\gamma^* \alpha} & \gamma^* S \end{array}$$

In alternative notation, this corresponds to $h = \{Y_i \xrightarrow{h_i} H_{\alpha(i)}\}_{i \in J}$.

We also say that \mathcal{Y} is a *refinement* of \mathcal{H} .

Assumption. We shall assume always that the components of the families are non empty, $H_i \neq \emptyset$ for all $i \in I$.

3.1. Definition. A *simplicial family* is a 3-tuple $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$, where H_\bullet, S_\bullet are simplicial objects in \mathcal{E}, \mathcal{S} respectively, and $H_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(S_\bullet)$ is a morphism of simplicial objects in \mathcal{E} . Remark that we assume that $(H_n)_w \neq \emptyset$.

A *morphism of simplicial families* $(Y_\bullet, J_\bullet, \zeta_\bullet) \xrightarrow{(h_\bullet, \alpha_\bullet)} (H_\bullet, S_\bullet, \zeta_\bullet)$, is a pair $Y_\bullet \xrightarrow{h_\bullet} H_\bullet$, $J_\bullet \xrightarrow{\alpha_\bullet} S_\bullet$, of simplicial morphisms making the following square commutative:

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{h_\bullet} & H_\bullet \\ \downarrow \xi_\bullet & & \downarrow \zeta_\bullet \\ \gamma^* J_\bullet & \xrightarrow{\gamma^* \alpha_\bullet} & \gamma^* S_\bullet \end{array}$$

We also say that \mathcal{Y}_\bullet is a *refinement* of \mathcal{H}_\bullet .

In alternative notation, $h_n = \{(Y_n)_w \xrightarrow{(h_n)_w} (H_n)_{\alpha_n(w)}\}_{w \in J_n}$.

¹Often called the *nerve* of \mathcal{U}

Notice that faces and degeneracy operators are morphisms of families:

$$\begin{array}{ccc}
 H_n & \begin{array}{c} \xrightarrow{d_i} \\ \xleftarrow{s_i} \end{array} & H_{n-1} \\
 \downarrow \zeta_n & \gamma^* \partial_i & \downarrow \zeta_{n-1} \\
 \gamma^* S_n & \begin{array}{c} \xrightarrow{\gamma^* \sigma_i} \\ \xleftarrow{\gamma^* \sigma_i} \end{array} & \gamma^* S_{n-1}
 \end{array}$$

and that in alternative notation correspond to families of maps

$$\{ (H_n)_w \xrightarrow{(d_i)_w} (H_{n-1})_{\partial_i(w)} \}_{w \in S_n} \quad \{ (H_n)_{\sigma_i(w)} \xleftarrow{(s_i)_w} (H_{n-1})_w \}_{w \in S_{n-1}}$$

We make now some considerations involving the first three terms.

$$\begin{array}{ccccc}
 & \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \end{array} & & \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \end{array} & \\
 H_2 & \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xleftarrow{d_2} \end{array} & H_1 & \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{array} & H_0 \\
 \downarrow \zeta_2 & & \downarrow \zeta_1 & & \downarrow \zeta_0 \\
 \gamma^* S_2 & \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\partial_1} \\ \xleftarrow{\sigma_1} \\ \xleftarrow{\partial_2} \end{array} & \gamma^* S_1 & \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\partial_1} \end{array} & \gamma^* S_0
 \end{array}$$

3.2. Remark.

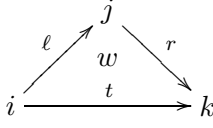
- (1) Each 1-simplex $\ell \in S_1$, $i \xrightarrow{\ell} j$ determines a 1-span

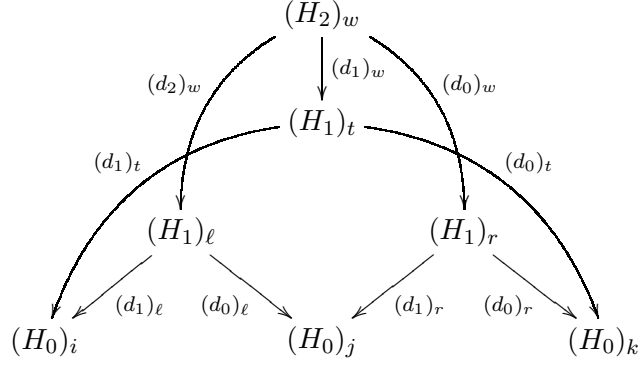
$$\begin{array}{ccc}
 & (H_1)_\ell & \\
 (d_1)_\ell \swarrow & & \searrow (d_0)_\ell \\
 (H_0)_i & & (H_0)_j
 \end{array}$$

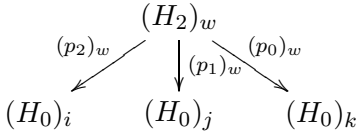
- (2) For each vertex $i \in S_0$ we have a morphism of spans:

$$\begin{array}{ccccc}
 & & (H_0)_i & & \\
 & \swarrow id & \downarrow (s_0)_i & \searrow id & \\
 (H_0)_i & & (H_1)_{\sigma_0(i)} & & (H_0)_i \\
 & \swarrow (d_1)_{\sigma_0(i)} & & \searrow (d_0)_{\sigma_0(i)} &
 \end{array}$$

(3)

Each 2-simplex $w \in S_2$  determines a 2-span



The composites determine three maps 

□

3.3. Definition. A self-dual simplicial family is a simplicial family $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$ together with a pair of strict self-dualities τ such that $\zeta_\bullet \circ \tau_\bullet = \gamma^* \tau_\bullet \circ \zeta_\bullet$. That is, for $w \in S_n$, $(H_n)_w \xrightarrow{\tau_n} (H_n)_{w^{op}}$.

$$\begin{array}{ccc}
 H_n & \xrightarrow{d_{n-i}} & H_{n-1} \\
 \downarrow \tau_n & & \downarrow \tau_{n-1} \\
 H_n & \xrightarrow{d_i} & H_{n-1}
 \end{array}
 \quad
 \begin{array}{ccc}
 (H_n)_w & \xrightarrow{(d_{n-i})_w} & (H_{n-1})_{\partial_{n-i}(w)} \\
 \downarrow (\tau_n)_w & & \downarrow (\tau_{n-1})_{\partial_{n-i}(w)} \\
 (H_n)_{w^{op}} & \xrightarrow{(d_i)_{w^{op}}} & (H_{n-1})_{\partial_{n-i}(w)^{op}}
 \end{array}$$

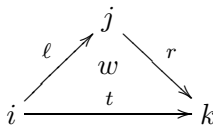
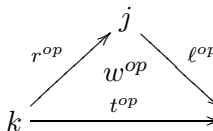
$\partial_{n-i}(w)^{op} = \partial_i(w^{op})$. This means that τ establishes an isomorphism between the span of w^{op} and the dual span of the span of w .

3.4. Remark.

- (1) For $\ell \in S_1$, if $i \xrightarrow{\ell} j$, then $j \xrightarrow{\ell^{op}} i$, and $(H_1)_{\ell^{op}} \cong (H_1)_\ell^{op}$ in the sense that $(\tau_1)_\ell$ establishes an isomorphism between the span ℓ^{op} and the dual span of ℓ :

$$\begin{array}{ccccc}
 & (d_1)_{\ell^{op}} & (H_1)_{\ell^{op}} & (d_0)_{\ell^{op}} & \\
 & \swarrow & \downarrow (\tau_1)_\ell & \searrow & \\
 (H_0)_j & & & & (H_0)_i \\
 & \nwarrow & \downarrow (H_1)_\ell & \nearrow & \\
 & (d_0)_\ell & & (d_1)_\ell &
 \end{array}$$

(2)

For $w \in S_2$, if  then  and

$(H_2)_{w^{op}} \cong (H_2)_w^{op}$ in the sense that that $(\tau_2)_w$ and $(\tau_1)_l, (\tau_1)_t, (\tau_1)_r$ establish an isomorphism between the 2-span of w^{op} and the dual 2-span of w . We leave to the interested reader to draw the corresponding diagram. \square

3.5. Example (Cech nerve simplicial family). Consider a family $U \xrightarrow{\zeta} \gamma^* I$ in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, and the Cech simplicial set N_\bullet , see example 2.6.

The *canonical simplicial object* U_\bullet is the simplicial object in \mathcal{E} whose n -simplexes are given by $U_n = U \times U \times \cdots \times U$ ($n+1$ times), $U_n = \sum_{(i_0, i_1, \dots, i_n) \in N_n} U_{i_0} \times U_{i_1} \cdots \times U_{i_n}$, in particular

$$U_0 = \sum_{i \in N_0} U_i, \quad U_1 = \sum_{(i, j) \in N_1} U_i \times U_j, \quad U_2 = \sum_{(i, j, k) \in N_2} U_i \times U_j \times U_k$$

It is easy to see that U_\bullet is a self-dual simplicial object with faces given by the appropriate projections and degeneracies by the appropriate diagonals. As for the self-duality, τ_1 is the usual symmetry of the cartesian product, τ_2 permutes the first and third factors, and leave unchanged the middle one, etc. The map $U \xrightarrow{\zeta} \gamma^* I$ determines a self-dual simplicial family $U_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(N_\bullet)$ that we call *Cech simplicial family*. \square

3.6. Proposition. *Given any simplicial family $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$, there is a canonical morphism of simplicial families*

$$\begin{array}{ccc} H_\bullet & \xrightarrow{h_\bullet} & U_\bullet \\ \downarrow \zeta_\bullet & & \downarrow \zeta_\bullet \\ \gamma^* S_\bullet & \xrightarrow{\gamma^* \alpha_\bullet} & \gamma^* N_\bullet \end{array}$$

where $(U \xrightarrow{\zeta} \gamma^* I) = (H_0 \xrightarrow{\zeta_0} \gamma^* S_0)$. If the family is self-dual, h and α commute with the dualities.

Proof. For the first three terms the proof should be clear by the remark 3.2: Given $\ell \in S_1$ and $w \in S_2$, $\alpha_1(\ell) = (\partial_1(\ell), \partial_0(\ell))$, $(h_1)_\ell = ((d_1)_\ell, (d_0)_\ell)$, $\alpha_2(w) = (\varrho_2(w), \varrho_1(w), \varrho_0(w))$, $(h_2)_w = ((p_2)_w, (p_1)_w, (p_0)_w)$. The second assertion follows by remark 3.4. For the higher simplexes the proof is the same and the interested reader can deduce the necessary calculations. \square

4. FAMILY HYPERCOVERS

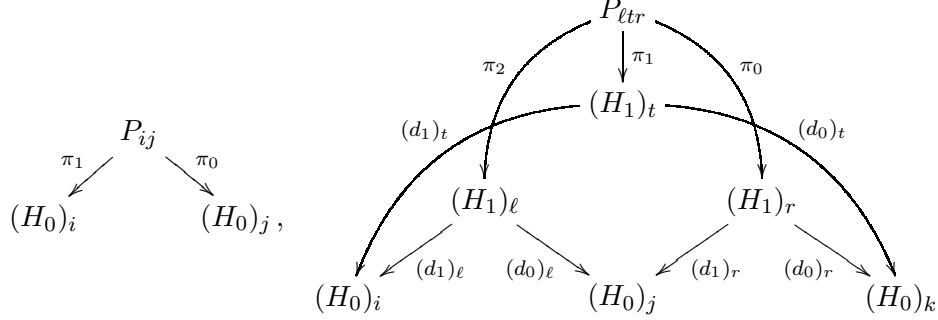
We consider now $U \xrightarrow{\zeta} \gamma^* I$ to be a cover, that is the map $U \rightarrow 1$ an epimorphism, and we will establish the condition that says that $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ is a hypercover in the sense of [2].

4.1 (The coskeleton).

By construction of the coskeleton we have, for $(\ell, t, r) \in S_1 \times S_1 \times S_1$,

$$\begin{array}{c} \begin{array}{ccc} & j & \\ \ell \nearrow & & \searrow r \\ i & \xrightarrow{t} & k \end{array} \quad (\text{no } w \text{ filling the triangle}) \\ (cosk_1 S_\bullet)_2 = \{(\ell, t, r) \mid i \xrightarrow{t} k \text{ (no } w \text{ filling the triangle)}\} \\ = \{(\ell, t, r) \mid \partial_0(\ell) = \partial_1(r) = j, \partial_1(t) = \partial_1(\ell) = i, \partial_0(t) = \partial_0(r) = k\} \end{array}$$

Let P_{ij} and $P_{\ell tr}$ be limit cones as follows:



$P_{ij} = (H_0)_i \times (H_0)_j$, $N_1 = \{(i, j) \mid P_{ij} \neq \emptyset\}$. Let T_2 be $T_2 \subset (\text{cosk}_1 S_\bullet)_2$, $T_2 = \{(\ell, t, r) \mid P_{\ell tr} \neq \emptyset\}$. It follows $(i, j, k) \in N_2$, thus there is a function $T_2 \rightarrow N_2$. Clearly for $w \in S_2$, $(\partial_2(w), \partial_1(w), \partial_0(w)) \in T_2$, defining a function $S_2 \rightarrow T_2$. By construction of the coeskeleton, we have:

$$(H_0)_i = U_i, \text{ indexed by } S_0,$$

$$((\text{cosk}_0 H_\bullet)_1)_{ij} = P_{ij}, \text{ indexed by } N_1,$$

$$((\text{cosk}_1 H_\bullet)_2)_{\ell tr} = P_{\ell tr}, \text{ indexed by } T_2.$$

$$H_0 = U_0, \quad (\text{cosk}_0 H_\bullet)_1 = U_1 = \sum_{(i, j) \in N_1} P_{ij},$$

$$(\text{cosk}_1 H_\bullet)_2 = \sum_{(\ell, t, r) \in T_2} P_{\ell tr} = \sum_{(i, j, k) \in N_2} \sum_{(\ell, t, r) \in (T_2)_{ijk}} P_{\ell tr}.$$

Clearly, for each $\ell \in S_1$, $i = \partial_1(\ell)$, $j = \partial_0(\ell)$, there is a map $(H_1)_\ell \rightarrow P_{ij}$, and for each $w \in S_2$, $\ell = \partial_2(w)$, $t = \partial_1(w)$, $r = \partial_0(w)$, there is a map $(H_2)_w \rightarrow P_{\ell tr}$, which are the components of the maps $H_1 \rightarrow (\text{cosk}_0 H_\bullet)_1$ and $H_2 \rightarrow (\text{cosk}_1 H_\bullet)_2$. From these considerations it follows:

4.2. Definition. A indexed hypercover refinement of a cover $(U \xrightarrow{\zeta} \gamma^* I)$ is a simplicial family $H_\bullet \xrightarrow{\zeta} S_\bullet$, $S_0 = I$, $H_0 = U$ such that the canonical morphism $H_\bullet \rightarrow U_\bullet$ is a hypercover, that is, the maps $H_k \rightarrow (\text{cosk}_{k-1} H_\bullet)_k$ are epimorphic ([2]). This is the case when for each $(i, j) \in N_1$ and $(\ell, t, r) \in T_2$, the families $\{(H_1)_\ell \rightarrow P_{ij}\}_{\ell \in (S_1)_{ij}}$ and $\{(H_2)_w \rightarrow P_{\ell tr}\}_{w \in (S_2)_{\ell tr}}$ are epimorphic. \square

As we have seen in remark 3.2, a simplicial family $H_\bullet \xrightarrow{\zeta} S_\bullet$ determines a collection of spans in each dimension. The n-spans in this collection are in one to one correspondence with the set S_n of n-simplexes of the index simplicial set, while the object H_n of the simplicial object is the coproduct of the vertices of all the n-spans indexed by S_n . This collection of spans conform a set of data which is equivalent to the simplicial family. Thus, we can determine a simplicial family by specifying a suitable collections of spans.

Consider any family $U \xrightarrow{\zeta} \gamma^* I$. We will construct self-dual simplicial refinements of the Čech simplicial family

$$\begin{array}{ccc} H_\bullet & \xrightarrow{h_\bullet} & U_\bullet \\ \downarrow \xi_\bullet & & \downarrow \zeta_\bullet \\ \gamma^* S_\bullet & \xrightarrow{\gamma^* \alpha_\bullet} & \gamma^* N_\bullet \end{array}$$

determined by a given set of spans. This is similar to the construction of the coskeleton functor.

4.3. Construction (0-Span refinements of the Čech simplicial family).

Let \mathcal{C} be a set of *non empty* objects closed under isomorphism and such that $U_i \in \mathcal{C}$, all i . We will construct a self-dual simplicial refinement such that for any $w \in S_n$, the components $(H_n)_w$ are in \mathcal{C} . We describe in detail the first three terms, where the procedure is best understood.

The simplicial set S_\bullet :

- (1) $S_0 = N_0$, $h_0 = id$.
- (2) S_1 is the set of all 1-spans with vertex in \mathcal{C} over the objects U_i . We write $S_1 \xrightarrow{\alpha_1} N_1$, and for $(i, j) \in N_1$, define the fibers of α_1 as:

$$(S_1)_{ij} = \{ \ell = U_i \begin{array}{c} \swarrow u \\ V \\ \searrow v \end{array} U_j, \quad V \in \mathcal{C} \}$$

and $\partial_0(\ell) = j$, $\partial_1(\ell) = i$, $\sigma_0(i) = (U_i \xleftarrow{id} U_i \xrightarrow{id} U_i)$.

- (3) S_2 is the set of all 2-spans over the objects U_i determined by the objects of \mathcal{C} . We write $S_2 \xrightarrow{\alpha_2} N_2$, and for $(i, j, k) \in N_2$, define the fibers of α_2 as:

$$(S_2)_{ijk} = \{ w = \begin{array}{ccccc} & & W & & \\ & x & \downarrow y & z & \\ & & Y & & \\ u_b \swarrow & & & & \searrow v_b \\ X & & & & Z \\ u_a \swarrow & & & & \searrow v_c \\ U_i & & U_j & & U_k \end{array}, \quad X, Y, Z, W \in \mathcal{C} \}$$

$$\begin{array}{ccc} & W & \\ f \swarrow & \downarrow h & \searrow g \\ U_i & U_j & U_k \end{array}$$

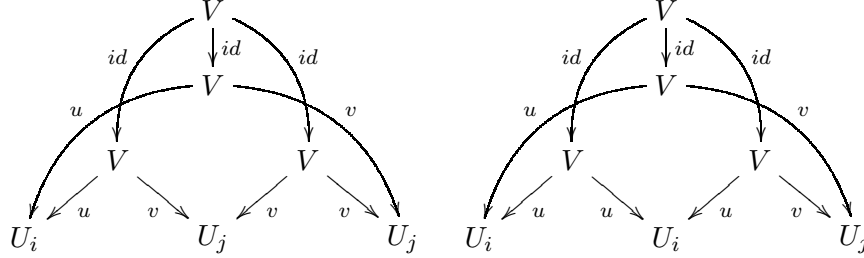
Taking the respective composites we have three maps $U_i \rightarrow U_j \rightarrow U_k$.

The face operators are clear:

$$\partial_2(w) = U_i \begin{array}{c} \swarrow u_a \\ X \\ \searrow v_a \end{array} U_j, \quad \partial_1(w) = U_i \begin{array}{c} \swarrow u_b \\ Y \\ \searrow v_b \end{array} U_k, \quad \partial_0(w) = U_j \begin{array}{c} \swarrow u_c \\ Z \\ \searrow v_c \end{array} U_k$$

Since $W \neq \emptyset$, $(\partial_2(w), \partial_1(w), \partial_0(w)) \in T_2$ (see 4.1), $S_2 \xrightarrow{\alpha_2} N_2$ factors $S_2 \rightarrow T_2 \rightarrow N_2$.

Given $\ell \in S_1$, the degeneracy operators $\sigma_0(\ell)$ and $\sigma_1(\ell)$ are given by:



The simplicial object H_\bullet :

It is determined by talking the coproducts of the vertices of the spans:

- (1) $H_0 \xrightarrow{\zeta_0} \gamma^* S_0$ is defined by:

$$(H_0)_i = U_i, \quad H_0 = \sum_{i \in S_0} U_i$$

- (2) $H_1 \xrightarrow{\xi_1} S_1$ is defined by:

$$(H_1)_\ell = V, \quad H_1 = \sum_{\ell \in S_1} V = \sum_{(i,j) \in N_1} \sum_{\ell \in (S_1)_{ij}} V.$$

Then $(s_0)_i = id_{U_i}$, $(d_0)_\ell = v$, and $(d_1)_\ell = u$. The map $V \xrightarrow{(u,v)} U_i \times U_j$ induce a map $H_1 \xrightarrow{h_1} U_1$ commuting with the simplicial structure.

- (3) $H_2 \xrightarrow{\xi_2} S_2$ is defined by:

$$\begin{aligned} (H_2)_w &= W, \quad H_2 = \sum_{w \in S_2} W = \sum_{(\ell, t, r) \in T_2} \sum_{w \in (S_2)_{\ell t r}} W \\ &= \sum_{(i, j, k) \in N_2} \sum_{(\ell, t, r) \in (T_2)_{ijk}} \sum_{w \in (S_2)_{\ell t r}} W. \end{aligned}$$

Define $(s_0)_\ell = id_V$, $(s_1)_\ell = id_V$, $(d_0)_w = z$, $(d_1)_w = y$, $(d_2)_w = x$. The map $W \xrightarrow{(f, g, h)} U_i \times U_j \times U_k$ induce a map $H_2 \xrightarrow{h_2} U_2$ commuting with the simplicial structure.

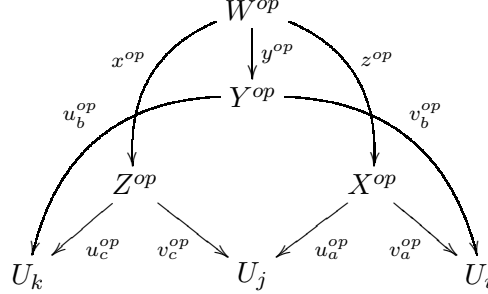
The self-duality τ :

Recall that $\tau_0 = id$. Given $\ell \in S_1$, we define $\tau_1(\ell) = \ell^{op}$ as the pullback on the right below:

$$\begin{array}{ccc} & V^{op} & \\ u^{op} \swarrow & & \searrow v^{op} \\ \ell^{op} = U_j & & U_i \end{array} \quad \begin{array}{ccc} V^{op} & \xrightarrow{(u^{op}, v^{op})} & U_j \times U_i \\ \downarrow (\tau_1)_\ell & & \downarrow \tau_1 \\ V & \xrightarrow{(u, v)} & U_i \times U_j \end{array}$$

It follows $u^{op} = v \circ (\tau_1)_\ell$ and $v^{op} = u \circ (\tau_1)_\ell$, so that $(\tau_1)_\ell$ establishes an isomorphism between the span ℓ^{op} and the dual span of ℓ . This shows that τ_1 commutes with the family structure, $\xi_1 \circ \tau_1 = \gamma^*(\tau_1) \circ \xi_1$ (see 3.3).

Given $w \in S_2$, we define $\tau_2(w) = w^{op}$ as the following 2-span:



Where W^{op} is defined as the pullback on the right below:

$$\begin{array}{ccc}
 & W^{op} & \\
 f^{op} \swarrow & \downarrow h^{op} & \searrow g^{op} \\
 U_k & U_j & U_i,
 \end{array}
 \qquad
 \begin{array}{ccc}
 W^{op} & \xrightarrow{(f^{op}, h^{op}, g^{op})} & U_k \times U_j \times U_i \\
 \downarrow (\tau_2)_w & & \downarrow \tau_2 \\
 W & \xrightarrow{(f, h, g)} & U_i \times U_j \times U_k
 \end{array}$$

We have $f^{op} = g \circ (\tau_2)_w$, $h^{op} = h \circ (\tau_2)_w$ and $g^{op} = f \circ (\tau_2)_w$. It follows there are maps x^{op} , y^{op} , z^{op} (as indicated in the span diagram above) which satisfy the equations:

$$(\tau_1)_r \circ x^{op} = z \circ (\tau_2)_w, \quad (\tau_1)_t \circ y^{op} = y \circ (\tau_2)_w \quad \text{and} \quad (\tau_1)_\ell \circ z^{op} = x \circ (\tau_2)_w.$$

This shows that $(\tau_2)_w$ and $(\tau_1)_\ell$, $(\tau_1)_t$, $(\tau_1)_r$ establish an isomorphism between the 2-span w^{op} and the dual 2-span of w . That is, τ_2 commutes with the family structure, $\xi_2 \circ \tau_2 = \gamma^*(\tau_2) \circ \xi_2$ (see 3.3). \square

4.4. Construction (1-Span refinements of the Cech simplicial family).

Let \mathcal{C}_{sp} be a set of *non empty* 1-spans closed under isomorphisms, the dual span, such that $(U_i \xleftarrow{id} U_i \xrightarrow{id} U_i) \in \mathcal{C}_{sp}$, all i , and such that $U_i \xleftarrow{u} V \xrightarrow{v} U_i$, $U_j \xleftarrow{v} V \xrightarrow{u} U_j \in \mathcal{C}_{sp}$ for all $U_i \xleftarrow{u} V \xrightarrow{v} U_j \in \mathcal{C}_{sp}$. Let \mathcal{C} be the set of vertices of the spans in \mathcal{C}_{sp} . We will construct a self-dual simplicial refinement such that the set of 1-spans determined by the 1-simplexes is the set \mathcal{C}_{sp} (and for $w \in S_n$, the component $(H_n)_w$ is in \mathcal{C}).

With the notation in construction 4.3, the 0-term is the same than in 4.3. The set S_1 is just defined to be the set \mathcal{C}_{sp} with the same simplicial structure, and S_2 is also defined in the same way, but with the assumption that the three 1-spans with vertices X , Y , Z should be in \mathcal{C}_{sp} . The simplicial object H_\bullet is defined exactly as in 4.3, and from the fact that \mathcal{C}_{sp} is closed under the dual span and isomorphisms it follows that the definition of the selfduality τ in 4.3 also applies here. \square

From definition 4.2 we have:

4.5. Proposition. *Given a cover $U \xrightarrow{\zeta} \gamma^*I$, if for each $(i, j) \in N_1$ and $(\ell, t, r) \in T_2$ we have:*

- (1) *If \mathcal{C} as in construction 4.3 is such that the families of all maps $\{W \rightarrow P_{ij}\}_{W \in \mathcal{C}}$ and $\{W \rightarrow P_{\ell tr}\}_{W \in \mathcal{C}}$ are epimorphic. Or*
- (2) *If \mathcal{C}_{sp} and \mathcal{C} as in construction 4.4 are such that the families of all maps $\{W \xrightarrow{(u,v)} P_{ij}\}_{(u,v) \in \mathcal{C}_{sp}}$ and $\{W \rightarrow P_{\ell tr}\}_{W \in \mathcal{C}}$ are epimorphic.*

Then the span refinement $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ is a hypercover. \square

4.6. Example (Canonical hypercover refinement of the Čech simplicial family by connected objects).

We assume the topos (or the site) to be *locally connected* ([2], [8]). Consider a cover $\mathcal{U} = (U, S, \zeta)$, $U \xrightarrow{\zeta} \gamma^* I$ such that all the U_i are connected objects. Then, taking as \mathcal{C} any set of connected generators the construction 4.3 yields an hypercover refinement of the Čech simplicial family in which all the components are connected. \square

5. FUNDAMENTAL GROUPOID OF A SIMPLICIAL FAMILY

We will now associate a groupoid in \mathcal{S} to any self-dual simplicial family satisfying the following filling condition. We remark that this condition does not hold for the Čech simplicial family but it will hold for the span refinements.

5.1. Definition. Let $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$, τ , be any self-dual simplicial family, we say that condition *G* is satisfied if the following holds:

$$\text{For all } i \xrightarrow{\ell} j \in S_1 \text{ there exists a 2-simplex } w \in S_2, \quad \begin{array}{ccc} & j & \\ \ell \nearrow & w & \searrow \ell^{op} \\ i & \xrightarrow{\partial_1(w)} & i \end{array}$$

such that $(d_1)_{\partial_1(w)} = (d_0)_{\partial_1(w)}$.

5.2. Proposition. Any 0-span or 1-span simplicial refinement $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$ of a family $U \xrightarrow{\zeta} \gamma^* I$ as in constructions 4.3 or 4.4 satisfies condition *G*.

Proof. Let $\ell \in S_1$ be any 1-simplex, and let $w \in S_2$ be as follows:

$$\begin{array}{c} \ell = U_i \xrightarrow{u} V \xrightarrow{v} U_j \\ \\ w = \begin{array}{ccccc} & & V & & \\ & id \nearrow & \downarrow id & \searrow (\tau_1)_\ell & \\ & V & & V & \\ u \nearrow & & & & \searrow u \\ U_i & \xrightarrow{u} & V & \xrightarrow{v} & U_j \\ & \nwarrow u & \searrow v & \nwarrow u^{op} & \searrow v^{op} \\ & & U_j & & U_i \end{array} \end{array}$$

It is clear that w meets the requirements of condition *G*. \square

5.3. Proposition (G-Fundamental Groupoid of a simplicial family).

Let $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$, τ , be a self dual simplicial family satisfying condition *G*. Consider the fundamental category of the simplicial set S_\bullet (proposition 2.1). Add to the equivalence relation that defines the morphism the following pairs:

(2) $(i \xrightarrow{\ell} j) \sim (i \xrightarrow{t} j)$ if there is a morphism of spans:

$$\begin{array}{ccccc}
 & & (H_1)_\ell & & \\
 & \swarrow (d_0)_\ell & \downarrow & \searrow (d_1)_\ell & \\
 (H_0)_i & & & & (H_0)_j \\
 & \swarrow (d_0)_t & \downarrow & \searrow (d_1)_t & \\
 & & (H_1)_t & &
 \end{array}$$

Then, the resulting category is groupoid.

Proof. We have:

$$\begin{array}{ccccc}
 & & (H_1)_{\partial_1(w)} & & \\
 & \swarrow (d_1)_{\partial_1(w)} & \downarrow & \searrow (d_0)_{\partial_1(w)} & \\
 (H_0)_i & \xleftarrow{id} & (H_0)_i & \xrightarrow{id} & (H_0)_i \\
 & \swarrow (d_1)_{\sigma_0(i)} & \downarrow (s_0)_i & \searrow (d_0)_{\sigma_0(i)} & \\
 & & (H_1)_{\sigma_0(i)} & &
 \end{array}$$

This shows that $(\ell^{op} \ell) \sim \partial_1(w) \sim \sigma_0(i)$. We use the assumption in the 1-simplex ℓ^{op} to show $(\ell \ell^{op}) \sim \sigma_0(j)$. \square

The G -fundamental groupoid of a family does not coincide with the fundamental groupoid of the index simplicial set. Because of this we have added the letter G to the word *fundamental* for the groupoid constructed in proposition 5.3.

6. DESCENT

Let S_\bullet be a simplicial set, $S_0 = I$,

6.1. Definition. A S_\bullet -descent datum on a family indexed by I , $R \rightarrow I$, is an isomorphism in \mathcal{S}_I , and it consists of the following data:

For each 1-simplex $i \xrightarrow{\ell} j$ a bijection $R_i \xrightarrow{s_\ell} R_j$ such that:

- 1) For each vertex $i \in I = S_0$, $s_{\sigma_0(i)} = id_{R_i}$.
- 2) For each 2-simplex $w \in S_2$, $s_{\partial_1(w)} = s_{\partial_0(w)} \circ s_{\partial_2(w)}$.

Recall that a (left) action of a small category with set of objects I in a I -indexed family $R \rightarrow I$ is a (covariant) set-valued functor R , $R(i) = R_i$, and for $x \in R_i$, $\ell \cdot x = R(\ell)(x)$. We say that the action is by isomorphisms if $R(\ell)$ is a bijection for all ℓ . Actions by isomorphisms are the same thing that actions of the groupoid resulting by formally inverting all the arrows of the category. The following is straightforward. For the record:

6.2. Proposition. For any simplicial set S_\bullet , there is a one to one correspondence between S_\bullet -descent data and left actions by isomorphisms of the fundamental category, that is, actions of the fundamental groupoid. With the obvious definition of morphisms this bijection extends to an isomorphism of categories (and of topoi). \square

Let $H_\bullet \rightarrow S_\bullet$ be any simplicial family, $S_0 = I$, $H_0 = U$,

6.3. Definition. A H_\bullet -descent datum σ on an object $\gamma^*(R) \times_{\gamma^*(I)} U \rightarrow U$ (where $R \rightarrow I$ is a family indexed by I), is an isomorphism in $\mathcal{E}_{/U}$, and it consists of the following data:

For each 1-simplex $i \xrightarrow{\ell} j$ an isomorphism σ_ℓ :

$$\begin{array}{ccc} \gamma^*R_i \times (H_1)_\ell & \xrightarrow{\sigma_\ell} & \gamma^*R_j \times (H_1)_{\ell^{op}} \\ \downarrow & & \downarrow \\ (H_1)_\ell & \xrightarrow{(\tau_1)_\ell} & (H_1)_{\ell^{op}} \end{array}$$

Notice that σ_ℓ is completely determined by its first projection that we denote with the same letter $\gamma^*R_i \times (H_1)_\ell \xrightarrow{\sigma_\ell} \gamma^*R_j$, or $(H_1)_\ell \xrightarrow{\hat{\sigma}_\ell} \gamma^*R_j \gamma^*R_i$. The identity and cocycle conditions are:

- (1) For each $i \in S_0$, $y \in \gamma^*R_i$, $x \in (H_0)_i$:
 $\sigma_{\sigma_0(i)}(y, (s_0)_i(x)) = (y, (s_0)_i(x)) : \hat{\sigma}_{\sigma_0(i)}((s_0)_i(x)) = id_{\gamma^*R_i},$
- (2) For each $w \in S_2$, $y \in \gamma^*R_i$, $x \in (H_2)_w$:
 $\sigma_{\partial_0(w)}(\sigma_{\partial_2(w)}(y, (d_2)_w(x)), (d_0)_w(x)) = \sigma_{\partial_1(w)}(y, (d_1)_w(x)) :$
 $\hat{\sigma}_{\partial_0(w)}(d_0(x)) \circ \hat{\sigma}_{\partial_2(w)}(d_2(x)) = \hat{\sigma}_{\partial_1(w)}(d_1(x)),$

The equations above correspond to commutative diagrams, the letters x, y can be thought as internal variables, or simply as a way to indicate how to construct the diagram.

Recall now from [2] (proposition 10.3).

6.4. Proposition. Given a cover $U \xrightarrow{\zeta} \gamma^*I$, a simplicial hypercover refinement of the Čech simplicial family (see proposition 3.6).

$$\begin{array}{ccc} H_\bullet & \xrightarrow{h_\bullet} & U_\bullet \\ \downarrow \zeta_\bullet & & \downarrow \zeta_\bullet \\ \gamma^*S_\bullet & \xrightarrow{\gamma^*\alpha_\bullet} & \gamma^*N_\bullet \end{array}$$

and a family of sets $R \rightarrow I$ indexed by I , consider for all $i \xrightarrow{\ell} j \in S_1$, $(i, j) = \alpha_1(\ell)$, the following diagrams:

$$\begin{array}{ccc} \gamma^*R_i \times U_i \times U_j & \xrightarrow{\sigma_{j,i}} & \gamma^*R_j \times U_j \times U_i \\ \uparrow \gamma^*R_i \times (h_1)_\ell & & \uparrow \gamma^*R_j \times (h_1)_{\ell^{op}} \\ \gamma^*R_i \times (H_1)_\ell & \xrightarrow{\sigma_\ell} & \gamma^*R_j \times (H_1)_{\ell^{op}} \end{array} \quad \begin{array}{ccc} U_i \times U_j & & \\ (h_1)_\ell \uparrow & \searrow \hat{\sigma}_{j,i} & \\ (H_1)_\ell & \xrightarrow{\hat{\sigma}_\ell} & \gamma^*R_j \gamma^*R_i \end{array}$$

Then, composing with $(h_1)_\ell$, that is the correspondence $\sigma_{j,i} \mapsto \sigma_\ell$, where σ_ℓ is such that $\sigma_{j,i} \circ \gamma^*R_i \times (h_1)_\ell = \gamma^*R_j \times (h_1)_{\ell^{op}} \circ \sigma_\ell$ or $\hat{\sigma}_{j,i} \mapsto \hat{\sigma}_\ell = \hat{\sigma}_{j,i} \circ (h_1)_\ell$, induces a bijection between H_\bullet -descent data σ_ℓ and U_\bullet -descent data $\sigma_{j,i}$ on objects of the form $\gamma^*(R) \times_{\gamma^*(I)} U \rightarrow U$. This actually establishes an isomorphism of the respective categories.

Proof. We give only an sketch of the proof. The correspondence is injective since the family $\{(h_1)_\ell\}_{\ell \in (S_1)_{ij}}$ is epimorphic. Given a H_\bullet -descent datum σ_ℓ , it can be seen that the family $\{\hat{\sigma}_\ell\}_{\ell \in (S_1)_{ij}}$ is compatible, thus there

exists a unique $\widehat{\sigma}_{j,i}$ such that $\widehat{\sigma}_\ell = \widehat{\sigma}_{j,i} \circ (h_1)_\ell$ all $\ell \in (S_1)_{ij}$. The cocycle and identity equations follow the fact that the family (remark 3.2, 3.)

$$\{(H_2)_w \xrightarrow{((p_2)_w, (p_1)_w, (p_0)_w)} U_i \times U_j \times U_k\}_{w \in (S_2)_{ijk}} \text{ is epimorphic.} \quad \square$$

6.5. Remark. A S_\bullet -descent datum s_ℓ on a family $R \rightarrow I$ induces a H_\bullet -descent datum $\sigma_\ell = \gamma^*(s_\ell) \times (\tau_1)_\ell$ on the object $\gamma^*(R) \times_{\gamma^*(I)} U$. \square

6.6. We say that a S_\bullet -descent datum (as in definition 6.1) is *consistent* if for each pair of 1-simplexes $\ell, t \in S_1$ as in (2) proposition 5.3, $s_\ell = s_t$.

We have (compare with proposition 6.2):

6.7. Proposition. *Given any self dual simplicial family $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$ satisfying condition G, there is a one to one correspondence between consistent S_\bullet -descent data and left actions of the G-fundamental groupoid of the family (Proposition 5.3). With the obvious definition of morphisms this bijection extends to an isomorphism of categories (and of topoi).* \square

7. COVERING PROJECTIONS

We recall now the concept of *covering projection* introduced in [3], for details we refer the reader to this source. Consider a cover $\mathcal{U} = U \xrightarrow{\zeta} \gamma^*I$ in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, and the Čech simplicial family $U_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(N_\bullet)$ (example 3.5). A locally constant object is an object X together with a trivialization structure θ . This structure consists in a family of isomorphisms $\{\theta_i\}_{i \in I}$:

$$\begin{array}{ccc} \gamma^*R_i \times U_i & \xrightarrow{\theta_i} & X \times U_i \\ & \searrow & \swarrow \\ & U_i & \end{array}$$

where $R \rightarrow I$, $\{R_i\}_{i \in I}$ is a family of sets. These objects are constructed by descent (see [5]) on a U_\bullet -descent datum σ on an object in \mathcal{E}_U of the form $\gamma^*R \times_{\gamma^*I} U \rightarrow U$, $\{\gamma^*R_i \times U_i \rightarrow U_i\}_{i \in I}$ (definition 6.3). Such a descent datum consists of a family of isomorphisms, $\{\sigma_{j,i}\}_{(i,j) \in N_1}$:

$$\begin{array}{ccc} \gamma^*R_i \times U_i \times U_j & \xrightarrow{\sigma_{j,i}} & \gamma^*R_j \times U_j \times U_i \\ \downarrow & & \downarrow \\ U_i \times U_j & \xrightarrow{\tau} & U_j \times U_i \end{array}$$

satisfying the corresponding identity and cocycle conditions in definition 6.3. The relationship between the trivialization structure θ and the descent datum σ is given in the following commutative diagram:

$$\begin{array}{ccc} \gamma^*R_i \times U_i \times U_j & \xrightarrow{\sigma_{j,i}} & \gamma^*R_j \times U_j \times U_i \\ \downarrow \theta_i \times U_j & & \downarrow \theta_j \times U_i \\ X \times U_i \times U_j & \xrightarrow{X \times \tau} & X \times U_j \times U_i \end{array}$$

Let $X \sim (R \rightarrow I, \sigma)$ be a locally constant object determined by a U_\bullet -descent datum σ on a cover $U \rightarrow \gamma^*I$. Recall from [3] the following

7.1. Definition.

An action span for X is a span $\ell = U_i \xleftarrow{u} V \xrightarrow{v} U_j$,

with $V \neq \emptyset$, and such that there is a bijection $S_i \xrightarrow{s_\ell} S_j$ (necessarily unique) such that the following diagram commutes

$$\begin{array}{ccc} \gamma^* R_i \times U_i \times U_j & \xrightarrow{\sigma_{j,i}} & \gamma^* R_j \times U_j \times U_i \\ \uparrow \gamma^* S_i \times (u,v) & & \uparrow \gamma^* S_j \times (v,u) \\ \gamma^* R_i \times V & \xrightarrow{\gamma^* s_\ell \times V} & \gamma^* R_j \times V \end{array}$$

7.2. Remark. Given a morphism of non empty spans $\ell' \rightarrow \ell$, $V' \rightarrow V$, if ℓ is an action span, then so it is ℓ' , and $s_{\ell'} = s_\ell$. \square

7.3. Definition (Definition 2.12 [3]). We say that a locally constant object $X \sim (S \rightarrow I, \sigma)$ trivialized by a cover $U \rightarrow \gamma^* I$ is a covering projection if, for each $(i, j) \in N_1$, the family $V \xrightarrow{(u,v)} U_i \times U_j$ is an epimorphic family, where (V, u, v) ranges over all action spans. By remark 7.2 it is equivalent to restrict V to a site of definition.

Every span with connected vertex is an action span, thus we have:

7.4. Proposition. In a locally connected topos every locally constant object is a covering projection. \square

7.5. Proposition. Let $X \sim (S \rightarrow I, \sigma)$ be a locally constant object trivialized by a cover $U \rightarrow \gamma^* I$, let \mathcal{C}_{sp} be the set of all action spans with V in a site of definition, and \mathcal{C} be the set of vertices of the spans in \mathcal{C}_{sp} . Then:

- (1) The conditions in construction 4.4 are satisfied.
- (2) The map $H_2 \rightarrow (\text{cosk}_1 H_\bullet)_2$ is an epimorphism.
- (3) If X is a covering projection, then the map $H_1 \rightarrow (\text{cosk}_0 H_\bullet)_1$ is an epimorphism. Thus $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ is an hypercovering.

Proof. (1) Observe that the dual span of an action span is an action span with inverse bijection s_ℓ^{-1} . The other requirements follow from the descent identity condition (1) in definition 6.3 and remark 7.2. Recall that in this case $(s_0)_i$ is the diagonal of U_i .

We refer now to proposition 4.5, 2:

(2) From remark 7.2 it follows that for any map $\emptyset \neq V \rightarrow P_{\ell tr}$, V is the vertex of a (in fact three) action spans, thus it is in \mathcal{C} .

(3) It holds by definition of covering projection. \square

7.6. Remark. From remark 7.2 it follows that the simplicial family $H_\bullet \rightarrow \gamma^*(S_\bullet)$ is a sieve in the sense that given any $V \in \mathcal{C}$, $V \subset (H_1)_\ell$, there exists $t \in S_1$ such that $(H_1)_t = V$.

Finally, from remark 7.2 and propositions 5.2, 6.4 and 7.4 we have:

7.7. Theorem. Let $X \sim (R \rightarrow I, \sigma)$ be a covering projection trivialized by a cover $U \rightarrow \gamma^* I$. Then there exist a self-dual simplicial hypercover refinement

of the Čech simplicial family (see proposition 3.6)

$$\begin{array}{ccc} H_{\bullet} & \xrightarrow{h_{\bullet}} & U_{\bullet} \\ \downarrow \zeta_{\bullet} & & \downarrow \zeta_{\bullet} \\ \gamma^* S_{\bullet} & \xrightarrow{\gamma^* \alpha_{\bullet}} & \gamma^* N_{\bullet} \end{array}$$

satisfying condition G (definition 5.1), and a consistent (cf 6.6) S_{\bullet} -descent datum $\{s_{\ell}\}_{\ell \in S_1}$ on the family $R \rightarrow I$ such that the corresponding H_{\bullet} -descent datum σ_{ℓ} (proposition 6.4) is of the form $\sigma_{\ell} = \gamma^*(s_{\ell}) \times (\tau_1)_{\ell}$ (remark 6.5). Vice-versa, any such descent datum on a self-dual simplicial hypercover refinement of the Čech simplicial family determines a covering projection trivialized by the cover $U \rightarrow \gamma^* I$. \square

We will also say that the covering projection is trivialized by the hypercover $H_{\bullet} \rightarrow \gamma^*(S_{\bullet})$. With the notation in the theorem above, from proposition 6.7 we have:

7.8. Theorem. *Given a cover $U \rightarrow \gamma^* I$ and a self-dual simplicial hypercover refinement of the Čech simplicial family satisfying condition G , then the category of covering projections trivialized by a consistent S_{\bullet} -descent datum $\{s_{\ell}\}_{\ell \in S_1}$ on a family $R \rightarrow I$, is isomorphic to the category (topos) of left actions of the G -fundamental groupoid of the family (Proposition 5.3). \square*

In the case of a locally connected topos, taking into account construction 4.3, example 4.6 and proposition 7.4 it follows:

7.9. Theorem. *Given any locally connected topos \mathcal{E} , the statement in theorem 7.7 holds for any locally constant object $X \sim (R \rightarrow I, \sigma)$ trivialized by a cover $U \rightarrow \gamma^* I$. Thus X can be constructed by a S_{\bullet} -descent datum $\{s_{\ell}\}_{\ell \in S_1}$ on the family $R \rightarrow I$, where S_{\bullet} is the simplicial set constructed in 4.3 with any set of connected generators. \square*

8. FUNDAMENTAL PROGROUPOID OF A TOPOS

Given a cover \mathcal{U} , in [3] we have shown that the category $\mathcal{G}_{\mathcal{U}}$ of covering projections trivialized by \mathcal{U} is an atomic topos, and then (by an application of Theorem 1 in [6] VIII, 3.) it follows it is the classifying topos of a localic groupoid $\mathbf{G}_{\mathcal{U}}$, $\mathcal{G}_{\mathcal{U}} = \beta \mathbf{G}_{\mathcal{U}}$. There is a faithful (but not full) functor $\mathcal{G}_{\mathcal{U}} \rightarrow \mathcal{E}$, and in this way it is easy to construct the colimit (inside \mathcal{E}) of the categories $\mathcal{G}_{\mathcal{U}}$ indexed by the covers \mathcal{U} ; $\mathcal{G}_{\mathcal{U}} \rightarrow c\mathcal{G}(\mathcal{E}) \rightarrow \mathcal{E}$. The category $c\mathcal{G}(\mathcal{E})$ is the category whose objects are all covering projections. Taking small coproducts of covering projections determines a topos $\mathcal{G}(\mathcal{E})$ together with a faithful functor $\mathcal{G}(\mathcal{E}) \rightarrow \mathcal{E}$ inverse image of a geometric morphism.

Given a cover \mathcal{U} and a hypercover refinement $\mathcal{H} = H_{\bullet} \rightarrow \gamma^*(S_{\bullet})$ as in theorem 7.7, it follows from theorem 7.8 that the category $\mathcal{G}_{\mathcal{H}}$ of covering projections trivialized by \mathcal{H} is the classifying topos of an ordinary groupoid $\mathbf{G}_{\mathcal{H}}$ (the G -fundamental groupoid of the simplicial family) $\mathcal{G}_{\mathcal{H}} = \beta \mathbf{G}_{\mathcal{H}}$. There is a faithful (but not full) functor $\mathcal{G}_{\mathcal{H}} \rightarrow \mathcal{E}$, and in this way it is easy to construct the colimit (inside \mathcal{E}) of the categories $\mathcal{G}_{\mathcal{H}}$ indexed by the hypercovers \mathcal{H} ; $\mathcal{G}_{\mathcal{H}} \rightarrow c\mathcal{G}(\mathcal{E}) \rightarrow \mathcal{E}$. It follows from theorem 7.7 that

every covering projection is in this category, so the category $c\mathcal{G}(\mathcal{E})$ and the resulting topos $\mathcal{G}(\mathcal{E})$ are the same that the ones constructed with the covers.

In loc. cit. we have exhibit $\mathcal{G}(\mathcal{E})$ as the classifying topos of a progroupoid on localic groupoids $\pi_1(\mathcal{E}) = \{\mathbf{G}_U\}_U$ (indexed by the filtered poset of covering sieves U). Furthermore, we have proved that $\pi_1(\mathcal{E})$ is the fundamental progroupoid of the topos \mathcal{E} in the sense that it classifies torsors, that is, there is an equivalence of categories $proGrpd[\pi_1(\mathcal{E}), K] \cong TOP[\mathcal{E}, \beta K] \cong K\text{-Tors}(\mathcal{E})$.

From the results in this paper we can exhibit $\mathcal{G}(\mathcal{E})$ as the classifying topos of a progroupoid on ordinary groupoids $\pi_1(\mathcal{E}) = \{\mathbf{G}_H\}_H$ (indexed by the filtered poset of hypercover sieves H). In this way we furnish an explicit construction of the fundamental progroupoid $\pi(\mathcal{E})$ showing at the same time that it is a progroupoid of ordinary groupoids. The salient feature that distinguishes the case of locally connected topos is that in that case the progroupoid is *strict*, namely, the transition morphisms are surjective on triangles (composably onto in the sense of [7]), the topos $\mathcal{G}(\mathcal{E})$ is a Galois topos and the inverse image functor $\mathcal{G}(\mathcal{E}) \rightarrow \mathcal{E}$ is full.

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